Self-Dual Vertex Operator Superalgebras of Large Minimal Weight

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Abstract

The new general upper bound $\mu \leq \left[\frac{c}{24}\right] + 1$ for the minimal weight μ of a self-dual vertex operator superalgebra of central charge $c \neq 23\frac{1}{2}$ is proven. For central charges $c \leq 48$, further improved estimates are given and examples of vertex operator superalgebras with large minimal weight are discussed. We also study the case of vertex operator superalgebras with $N{=}1$ supersymmetry which was first considered by Witten in connection with three-dimensional quantum gravity. The upper bound $\mu^* \leq \frac{1}{2}\left[\frac{c}{12}\right] + \frac{1}{2}$ for the minimal superconformal weight is obtained for $c \neq 23\frac{1}{2}$.

In addition, we show that it is impossible that the monster sporadic group acts on an extremal self-dual $N{=}1$ supersymmetric vertex operator superalgebra of central charge 48 in a way proposed by Witten if certain standard assumptions about orbifold constructions hold. The same statement holds for extremal self-dual vertex operator algebras of central charge 48.

1 Introduction

Extremal self-dual vertex operator algebras and superalgebras have been defined in [Höh95], Chapter 5. Extremal refers here to the property that the degree of a Virasoro highest weight vector different from the vacuum vector must be larger then certain bounds obtained from conditions on the characters. The smallest such degree is called the minimal weight. For small values of the central charge c several examples with interesting automorphism are known, like the Moonshine module $V^\natural$ [FLM88] for $c = 24$ and the shorter Moonshine module $VB^\natural$ for $c = 23\frac{1}{2}$ [Höh95]. The notion extremal is analogous to similar ones for binary codes and for lattices. These two cases have been studied much more intensively because of their relations with more geometric problems and their applications to data processing and transmissions, cf., for example, [Sto99]. Examples of such codes and lattices are known for lengths respectively dimensions up to about 100.

Recently it was shown by the author that extremal vertex operator algebras can be used to construct conformal t-designs [Höh07], an algebraic structure sharing many

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properties with classical block designs and spherical designs. In another development, Witten proposed that extremal vertex operator algebras can be used to describe three-dimensional quantum gravity with negative cosmological constant [Wit07]. He also considers supergravity. Extremal vertex operator algebras have been further investigated in [Man07, GY07, Gab07, AKM07].

The present paper continuous the study of self-dual vertex operator superalgebras of large minimal weight initiated in [Höh95]. In Section 2, the previous upper bound $\mu \leq \frac{1}{2}\left[\frac{c}{8}\right] + \frac{1}{2}$ ([Höh95], Cor. 5.3.3) for the minimal weight μ of a self-dual vertex operator superalgebra is improved in Theorem 1 to

$$\mu \leq \left[\frac{c}{24}\right] + 1$$

for central charges $c \neq 23\frac{1}{2}$. This upper bound can sometimes be improved further; Table 1 lists our results for $c \leq 48$. For $c \leq 24$, there are always examples meeting the bound. For $24\frac{1}{2} \leq c \leq 48$, we discuss examples which yield the given lower bounds.

In Section 3, we study the minimal superconformal weight of a self-dual $N=1$ supersymmetric vertex operator superalgebra. We define the minimal superconformal weight μ^* as the smallest positive degree of a highest weight vector for the N=1 super Virasoro algebra. In Theorem 6 we obtain for $c \neq 23\frac{1}{2}$ the upper bound

$$\mu^* \leq \frac{1}{2}\left[\frac{c}{12}\right] + \frac{1}{2},$$

which for central charges divisible by 12 was found in [Wit07]. A self-dual $N=1$ supersymmetric vertex operator superalgebra with minimal superconformal weight meeting this bound is called extremal.

In the final section, we take a closer look on the case of central charge $c = 48$. The relation between self-dual vertex operator superalgebras of minimal weight 5/2 and extremal self-dual N=1 supersymmetric vertex operator superalgebras with extremal self-dual vertex operator algebras is studied. This allows us for $c = 48$ to conclude that under reasonable assumptions it is impossible for the monster simple group to act by automorphisms on an extremal self-dual vertex operator algebra or an extremal self-dual N=1 supersymmetric superalgebra.

It remains an open problem if self-dual vertex operator algebras or ($N=1$ supersymmetric) vertex operator superalgebras with minimal (superconformal) weight larger than 2 exist.

In the rest of the introduction, we give precise definitions and discuss the required results about vertex operator superalgebras. We assume that the reader is familiar with the general notation of vertex operator superalgebras.

The Virasoro algebra is the complex Lie algebra spanned by L_n, $n \in \mathbf{Z}$, and the central element C with Lie bracket

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\, C \tag{1}$$

where $\delta_{k,0} = 1$ if $k = 0$ and $\delta_{k,0} = 0$ otherwise. For a pair (c, h) of complex numbers the Verma module $M(c, h)$ is a representation of the Virasoro algebra generated by a

highest weight vector $v \in M(c,h)$ with $Cv = cv$, $L_0 v = hv$ and $L_n v = 0$ for $n \geq 1$. For $h = 0$, the module $M(c,0)$ has a quotient isomorphic to $M(c,0)/M(c,1)$.

We *assume* that the vertex operator algebras V in this paper are isomorphic to a direct sum of highest weight modules for the Virasoro algebra, i.e., one has

$$V = \bigoplus_{i \in I} M_i,$$

where each M_i is a quotient of a Verma modules $M(c,h)$ with $h \in \mathbf{Z}_{\geq 0}$. One has therefore a natural decomposition

$$V = \bigoplus_{h=0}^{\infty} \overline{M}(h) \tag{2}$$

where $\overline{M}(h)$ is a direct sum of finitely many quotients of the Verma module $M(c,h)$. The module $\overline{M}(0)$ is the vertex operator subalgebra of V generated by ω which we denote also by V_ω and is a quotient of $M(c,0)/M(c,1)$. The smallest $h > 0$ for which $\overline{M}(h) \neq 0$ was called in [Höh95] the *minimal weight* of V and denoted by $\mu(V)$. (If no such $h > 0$ exists, we let $\mu(V) = \infty$.)

A vertex operator algebra is called *rational* (cf. [DLM98]) if every admissible module is completely reducible. In this case there are only finitely many irreducible admissible modules up to isomorphism and every irreducible admissible module is an ordinary module. A vertex operator algebra is called *simple* if it is irreducible as a module over itself.

For an irreducible module W there exists an h such that $W = \bigoplus_{k \in \mathbf{Z}_{\geq 0}} W_{k+h}$ with $W_h \neq 0$, where the degree n subspace W_n is the eigenspace of L_0 for the eigenvalue n. We call h the *conformal weight* of the module W.

The *character* of a module W of conformal weight h is defined by

$$\chi_W = q^{-c/24} \sum_{k \in \mathbf{Z}_{\geq 0}} \dim W_{k+h} q^{k+h}.$$

If V is assumed to be rational and satisfying the C_2-*cofiniteness condition* $\dim(V/\mathrm{Span}\{x_{(-2)}y \mid x, y \in V\}) < \infty$ it is a result of Zhu [Zhu90] that χ_W is a holomorphic function on the complex upper half plane in the variable τ for $q = e^{2\pi i \tau}$. We assume in this paper that the C_2-cofiniteness condition is satisfied. The family $\{\chi_W\}_W$, where W runs through the isomorphism classes of irreducible V-modules W, transforms as a vector-valued modular function for the modular group $\mathrm{SL}_2(\mathbf{Z})$ acting on the upper half plane in the usual way.

A rational vertex operator algebra V is called *self-dual* (other authors use the notation holomorphic or meromorphic) if the only irreducible V-module is V itself. It follows from the above mentioned result of Zhu that the character χ_V is a weighted homogeneous polynomial of weight c in $\sqrt[3]{j}$ (given the weight 8) and 1 (weight 24) where $\sqrt[3]{j}$ is the third root of the elliptic modular function j (cf. [Höh95], Thm. 2.1.2). In particular, the central charge c of a self-dual vertex operator algebra is divisible by 8. It was shown in [Höh95], Cor. 5.2.3, that the minimal weight of a self-dual vertex operator algebra satisfies $\mu(V) \leq [c/24]+1$. A self-dual vertex operator algebra meeting this bound is called *extremal.*

We also need an *unitary* condition. Sufficient for this paper is to assume that that V has a real form with positive-definite invariant bilinear form and the conformal weights of all irreducible V-modules are nonnegative.

We call a self-dual vertex operator superalgebra $V = V_{(0)} \oplus V_{(1)}$ rational if its even vertex operator subalgebra $V_{(0)}$ is rational and has the same associated modular braided tensor category as the even vertex operator subalgebra of $V_{\text{Fermi}}^{\otimes c}$, where V_{Fermi} is the vertex operator superalgebra of central charge $\frac{1}{2}$ describing a single fermion. One has $c \in \frac{1}{2}\mathbf{Z}$ (see [Höh95], Thm. 2.2.2). The fusion algebra of $V_{(0)}$ is for $c \in 2\mathbf{Z}$ isomorphic to $\mathbf{Z}[\mathbf{Z}_2 \times \mathbf{Z}_2]$, for $c \in \mathbf{Z} \setminus 2\mathbf{Z}$ isomorphic to $\mathbf{Z}[\mathbf{Z}_4]$ and for $c \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$ isomorphic to the Ising fusion algebra ([Höh95], Thm. 2.2.5). The three (c non-integral) or four (c integral) types of irreducible $V_{(0)}$-modules $V_{(0)}$, $V_{(1)}$, $V_{(2)}$ (and $V_{(3)}$) have the conformal weights 0 (for $V_{(0)}$), $\frac{1}{2}$ (mod 1) (for $V_{(1)}$) and $c/8$ (mod 1) (for $V_{(2)}$ and, in case of four modules, $V_{(3)}$). The V-module $V_{(2)} \oplus V_{(3)}$ for integral c, respectively $V_{(2)}$ for c nonintegral, is called the *shadow* of V and denoted by V'.

We call two self-dual vertex operator algebras W and $\widetilde{W}$ neighbours if there exists a rational self-dual vertex operator superalgebra $V = V_{(0)} \oplus V_{(1)}$ such that $W = V_{(0)} \oplus V_{(2)}$ and $W = V_{(0)} \oplus V_{(3)}$ where $V_{(0)}$, $V_{(1)}$, $V_{(3)}$ and $V_{(3)}$ are the four irreducible $V_{(0)}$-modules as above. The pairs V and W as well as V and $\widetilde{W}$ are also called neighbours. For central charge c divisible by 8 a self-dual vertex operator superalgebra $V = V_{(0)} \oplus V_{(1)}$ has the two neighbour vertex operator algebras $V_{(0)} \oplus V_{(2)}$ and $V_{(0)} \oplus V_{(3)}$ which could be isomorphic.

A vertex operator superalgebra $V_{(0)} \oplus V_{(1)}$ comes with a natural involutive automorphism σ which acts by $+1$ on $V_{(0)}$ and -1 on $V_{(1)}$.

The character of a self-dual rational unitary vertex operator superalgebra has the form

$$\chi_V = \sum_{r=0}^{k} a_r \chi_{1/2}^{2c-24r} \tag{3}$$

where

$$\chi_{1/2} = q^{-1/48} \prod_{n=0}^{\infty} (1 + q^{n+1/2}), \tag{4}$$

$k = \left[\frac{c}{8}\right]$ and the $a_0, \ldots, a_k$ are uniquely determined integers ([Höh95], Thm. 2.2.3). The character of the shadow is

$$\chi_{V'} = \alpha \sum_{r=0}^{k} a_r \widetilde{\chi}_{1/2}^{2c-24r} \tag{5}$$

with $\alpha = 1$ for integral and $\alpha = 1/\sqrt{2}$ for nonintegral values of c and

$$\widetilde{\chi}_{1/2} = \sqrt{2} q^{1/24} \prod_{n=0}^{\infty} (1 + q^n). \tag{6}$$

If we let $q = e^{2\pi i \tau}$ with τ in the complex upper half-pane, then $\chi_{1/2}$, and hence χ_V, can be considered as the Fourier expansion of a modular function for the modular group

$\Gamma_\theta = \langle S, T^2\rangle$ in the cusp $i\infty$ where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The character $\chi_{V'}$ is then the expansion of $e^{2\pi i c/24}\chi_V$ (c integral), respectively $e^{2\pi i c/24}\chi_V/\sqrt{2}$ (c nonintegral), in the other cusp of $\overline{\mathbf{H}/\Gamma_\theta}$ represented by 1.

2 Self-dual vertex operator superalgebras

First we prove our new general upper bound for the minimal weight of a self-dual vertex operator superalgebra.

Theorem 1 *A self-dual vertex operator superalgebra V of central charge c has minimal weight*

$$\mu(V) \leq \left[\frac{c}{24}\right] + 1$$

unless $c = 23\frac{1}{2}$ in which case $\mu(V) \leq 3/2$.

Proof. We can assume that V is not a vertex operator algebra, i.e., $V_{(1)} \neq 0$, because for vertex operator algebras the estimate for $\mu(V)$ given in the theorem was obtained in [Höh95], Corollary 5.2.3.

As discussed in the introduction, the character of V is a Laurent polynomial in $\chi_{1/2}$ of the form

$$\chi_V = \sum_{r=0}^{k} a_r \chi_{1/2}^{2c-24r} = q^{-c/24} \sum_{n=0}^{\infty} C_n\, q^{n/2} \tag{7}$$

The character of the shadow equals

$$\chi_{V'} = \alpha \sum_{r=0}^{k} a_r \widetilde{\chi}_{1/2}^{2c-24r} = \alpha\, q^{c/12-[c/8]} \sum_{n=0}^{\infty} B_n\, q^n, \tag{8}$$

with $\alpha = 1$ for integral and $\alpha = 1/\sqrt{2}$ for nonintegral values of c.

First we consider the case $c < 32$. In the range $\frac{1}{2} \leq c < 24$, we have to show that $\mu(V) \leq 1$ unless $c = 23\frac{1}{2}$. From equation (7) one sees directly that for $\frac{1}{2} \leq c < 8$ one has $\mu(V) = \frac{1}{2}$. For $8 \leq c < 16$, the condition $C_0 = 1$ and $C_1 = 0$ determines $C_2 = \dim V_1$ which turns out to be positive (cf. Table 5.3 and Table 5.4 of [Höh95]). Hence $\mu(V) = 1$ as all vectors in V_1 are Virasoro highest weight vectors. For $16 \leq c < 23\frac{1}{2}$, the condition $C_0 = 1$ and $C_1 = C_2 = 0$ determines χ_V and thus $\chi_{V'}$. Unless $c = 23\frac{1}{2}$, the character $\chi_{V'}$ has nonintegral coefficients (cf. Table 5.4 of [Höh95]) and hence $\mu(V) > 1$ is impossible. For $c = 23\frac{1}{2}$ one gets $C_3 = \dim V_{3/2} = 4371$ and hence $\mu(V) = 3/2$ as again all vectors in $V_{3/2}$ are Virasoro highest weight vectors. Finally, for $24 \leq c < 31\frac{1}{2}$, we have to show that $\mu(V) \leq 2$. The condition $C_0 = 1$ and $C_1 = C_2 = C_3 = 0$ determines $C_4 = \dim V_2$. In all cases one has $C_4 > 1$ and hence $\mu(V) = 2$ as $C_4 - 1$ is the dimension of the space of Virasoro highest weight vectors.

For $c \geq 32$, the proof will only use that the coefficients B_i of the characters of the shadow V' are nonnegative rational numbers. Let $m = \left[\frac{c}{24}\right] + 1$ and suppose

that $\mu(V) > m$. We denote by $\chi_{M_c} = q^{-c/24}\prod_{n=2}^{\infty}(1-q^n)^{-1}$ the character of $V_\omega = M(c,0)/M(c,1)$. Then

$$\chi_V = \chi_{M_c}\cdot(1 + A_{2m+1}\,q^{m+1/2} + A_{2m+2}\,q^{m+1} + \cdots)$$

which determines the a_i for $0 \le i \le 2m$. We will show that $a_{2m} < 0$. On the other hand, equation (8) allows us to write a_{2m} as a linear combination of the B_i for $0 \le i \le k-2m$, say $a_{2m} = \sum_{i=0}^{k-2m}\beta_i B_i$. We will show that the β_i are all nonnegative, and thus $a_{2m} \ge 0$, a contradiction. Hence the assumption $\mu(V) > m$ must be wrong and $\mu(V) \le \left[\frac{c}{24}\right] + 1$.

To determine a_{2m}, we let $p = q^{1/2}$ and expand $\chi_{M_c}\cdot\chi_{1/2}^{-2c}$ in powers of $\phi = \chi_{1/2}^{-24} = p + O(p^2)$. We get

$$\chi_{M_c}\cdot\chi_{1/2}^{-2c} = \sum_{r=0}^{\infty}\alpha_r\phi^r,$$

where by the Bürmann-Lagrange theorem the coefficient α_r is given for $r > 0$ by the coefficient of p^{r-1} in

$$\frac{1}{r}\frac{d(\chi_{M_c}\cdot\chi_{1/2}^{-2c})}{dp}\left(\frac{p}{\phi}\right)^r = \frac{1}{r}\,p^r\cdot\chi_{1/2}^{24r-2c-1}\left[\chi'_{M_c}\chi_{1/2} - 2c\cdot\chi_{M_c}\chi'_{1/2}\right] \tag{9}$$

and $\alpha_0 = 1$. Since

$$\sum_{r=0}^{k}a_r\phi^r = \chi_V\cdot\chi_{M_c}^{-1}\cdot\chi_{M_c}\cdot\chi_{1/2}^{-2c} = \left(1 + \sum_{n=2m+1}^{\infty}A_np^n\right)\left(\sum_{r=0}^{2m}\alpha_r\phi^r + \sum_{r=2m+1}^{\infty}\alpha_r\phi^r\right),$$

comparing coefficients on both sides gives $a_r = \alpha_r$ for $0 \le r \le 2m$. Here we used that $2m = 2\,[c/24]{+}2 \le [c/8] = k$ for $c \ge 32$. The coefficients of $\chi_{1/2}^{24m-2c-1}$ are nonnegative because $24\cdot 2m - 2c - 1 = 48\,[c/24] - 2c + 47 \ge 0$. It follows from Lemma 2 below that all coefficients of $\chi'_{M_c}\chi_{1/2} - 2c\cdot\chi_{M_c}\chi'_{1/2}$ are negative for $c \ge 1.01$. Equation (9) gives now $a_{2m} < 0$.

For the second estimate of a_{2m}, we obtain from (8) the equation

$$\sum_{r=0}^{k}a_r(\widetilde{\chi}_{1/2}^{24})^{k-r} = q^{c/12-[c/8]}\widetilde{\chi}_{1/2}^{24k-2c}\sum_{n=0}^{\infty}B_n\,q^n. \tag{10}$$

Let

$$q^{n+c/12-[c/8]}\widetilde{\chi}_{1/2}^{24k-2c} = \sum_{r=0}^{\infty}\beta_{n,r}\,\widetilde{\phi}^r \tag{11}$$

be the expansion of $q^{c/12-[c/8]}\widetilde{\chi}_{1/2}^{24k-2c}\,q^n$ in powers of $\widetilde{\phi} = \widetilde{\chi}_{1/2}^{24} = 2^{12}q + O(q^2)$. Using again the Bürmann-Lagrange theorem, we have that $\beta_{n,r}$ is for $r > 0$ the coefficient of q^{r-1} in

$$\frac{1}{r}\frac{d(\widetilde{\chi}_{1/2}^{24k-2c}q^{n+c/12-[c/8]})}{dq}\left(\frac{q}{\widetilde{\phi}}\right)^r$$

$$= \frac{1}{r} q^{n+c/12-k-1+r} \widetilde{\chi}_{1/2}^{24k-2c-1-24r} \left[(24k-2c) q \widetilde{\chi}'_{1/2} + (n + \frac{c}{12} - k) \widetilde{\chi}_{1/2} \right]$$

and $\beta_{0,0} = 2^{24[c/24]-c+24}$. The coefficients of $\widetilde{\chi}_{1/2}^{24k-2c-1-24r}$ for $r = k - 2m$ are nonnegative because $24k - 2c - 1 - 24r = 48\,[c/24] - 2c + 47 \geq 0$. With $\widetilde{\chi}_{1/2}$ also $(24k-2c)q\widetilde{\chi}'_{1/2} + (n + \frac{c}{12} - k)\widetilde{\chi}_{1/2}$ has for all $n \geq 0$ positive coefficients. Thus $\beta_{n,k-2m}$ is positive.

Comparing equation (10) and (11) gives $a_{k-r} = \sum_{n=0}^{r} \beta_{n,r} B_n$ and hence $a_{2m} \geq 0$, the desired contradiction. □

Remark: The analogous result for unimodular lattices was obtained by Rains and Sloane [RS98]. For even self-dual binary codes and even Kleinian codes the corresponding results can be found in [Rai98].

Lemma 2 *The coefficients of the series $2c\chi_{M_c}\chi'_{1/2} - \chi'_{M_c}\chi_{1/2}$ are all positive for $c \geq 1.01$.*

Proof. Let $A = p^{1/24}\chi_{1/2} = \prod_{n=0}^{\infty}(1 + p^{2n+1})$ and $B = p^{c/12}\chi_{M_c} = \prod_{n=2}^{\infty}(1 - p^{2n})^{-1}$. Then

$$2c\chi_{M_c}\chi'_{1/2} - \chi'_{M_c}\chi_{1/2} = p^{-c/12-1/24}(2cBA' - B'A) = p^{-c/12-1/24} B(2cA' - A \cdot B'/B)$$

and it is enough to show that $2cA' - A \cdot B'/B$ has positive coefficients. One has

$$\begin{aligned} B'/B &= \frac{d}{dp}\left(\sum_{n=2}^{\infty} -\log(1 - p^{2n})\right) \\ &= \sum_{n=2}^{\infty} \frac{2n\, p^{2n-1}}{1 - p^{2n}} \\ &= \sum_{n=2}^{\infty} 2(\sigma(n) - 1) p^{2n-1}, \end{aligned}$$

where $\sigma(n)$ is the sum of the positive divisors of n. Thus the coefficients of B'/B can be estimated from above by the coefficients of $2\sum_{n=1}^{\infty}\sigma(n)p^{2n-1}$. Let $A = \sum_{n=0}^{\infty} f(n)p^n$. Then the coefficient of p^n in AB'/B is estimated from above by 2 times

$$\begin{aligned} \sum_{k=1}^{[\frac{n+1}{2}]} \sigma(k) f(n - (2k-1)) &= \sum_{k=1}^{[\frac{n+1}{2}]} f(n+1-2k) \sum_{d|k} \frac{k}{d} \\ &= \sum_{d=1}^{[\frac{n+1}{2}]} \frac{1}{d} \sum_{\substack{k \leq [\frac{n+1}{2}] \\ d|k}} f(n+1-2k)k \\ &= \sum_{d=1}^{[\frac{n+1}{2}]} \sum_{r=1}^{[[\frac{n+1}{2}]/d]} f(n+1-2rd) \cdot r \\ &\sim \sum_{d=1}^{[\frac{n+1}{2}]} \int_0^{n/2d} f(n - 2td) \cdot t \cdot dt \end{aligned}$$

$$= \sum_{d=1}^{[\frac{n+1}{2}]} \frac{1}{d^2} \int_0^{n/2} f(n-2x)\cdot x \cdot dx$$
$$< \frac{\pi^2}{6} \cdot \int_0^{n/2} f(n-2x)\cdot x\cdot dx. \qquad (12)$$

The coefficient $f(n)$ counts the number of partitions of n into odd and unequal parts and one has for $n \longrightarrow \infty$ the asymptotic formula ([Hag64], Corollary of Thm. 6)

$$f(n) = \sqrt{6}\,(24n-1)^{-3/4} \exp\left(\pi\sqrt{24n-1}/12\right)\left(1+\mathrm{O}(n^{-1/2})\right).$$

Using this approximation for $f(x)$, the integral $\int_0^{n/2} f(n-2x)\cdot x\cdot dx$ can be evaluated explicitly and one obtains $\lim_{n\longrightarrow\infty}\left(\int_0^{n/2} f(n-2x)\cdot x\cdot dx\right)/(nf(n)) = 6/\pi^2$. Hence the coefficients of AB'/B are smaller than the coefficients $(n+1)f(n+1)$ in $2cA'$ for large n.

It is now also straightforward to justify the approximation of the sum by the integral in (12): The function $f(n-2t)\cdot t$ is not monotone on $[1, n/2]$ but has a single maximum at $t_0 \sim \sqrt{24}/(2\pi)\sqrt{n}$. The possible approximation error is therefore not larger then $\sum_{d=1}^{(n+1)/2} \frac{1}{d}\cdot f(n-2t_0)t_0 \sim \log(n/2) f(n-2t_0)t_0$. But

$$\lim_{n\longrightarrow\infty} \log(n/2) f(n-2t_0)t_0/(nf(n)) = 0.$$

We skip the explicit computation of an N_0 such that the Lemma holds for $n \geq N_0$. For $n < 3000$ we checked the Lemma directly. $\square$

For smaller values of the central charge c, one can often improve the general upper bound of Theorem 1. Table 1 lists our results for $c \leq 48$.

We use that the character of a vertex operator superalgebra and of its shadow must have nonnegative integral coefficients. (More precisely, the dimensions of the Virasoro primaries must be nonnegative.)

As an example, assume that a vertex operator superalgebra of central charge $c = 33\frac{1}{2}$ and minimal weight 2 exists. This would imply $a_0 = 1$, $a_1 = -67$, $a_2 = 670$, $a_3 = -201$ and hence

$$\begin{aligned}\chi_V &= q^{-33\frac{1}{2}/24}\left(1 + (56816 + a_4)\, q^2 + (2072444 - 29a_4)\, q^{5/2} + \cdots\right),\\ \chi_{V'} &= q^{-33\frac{1}{2}/24}\left(a_4/32768\, q^{3/16} + (823296 - 29a_4)/32768\, q^{19/16} + \cdots\right).\end{aligned}$$

The initial term of $\chi_{V'}$ gives $a_4 \geq 0$ and $2^{15} \mid a_4$. The second coefficient of $\chi_{V'}$ is only nonnegative for $a_4 = 0$ in which case it is nonintegral.

This kind of arguments gives the upper bounds for all central charges c listed besides for $c = 10$, 11, $12\frac{1}{2}$, 13, $13\frac{1}{2}$, $14\frac{1}{2}$ and $16\frac{1}{2}$, for which we used that the list of vertex operator superalgebras given in Thm. 5.3.2 [Höh95] is complete, which in turn depends on Schellekens classification [Sch93] of self-dual vertex operator algebras of central charge 24. However, the method of Schellekens can also directly be applied for $c \leq 16\frac{1}{2}$ to show that the obtained value for $\dim V_1$ cannot be realized.

It is known that any unimodular lattice in dimensions $n \equiv 0 \pmod{24}$ meeting the upper bound analogous to Theorem 1 must be even (see [Gau01]). A similar result holds for even self-dual binary codes (cf. [RS98]). For vertex operator superalgebras of small central charges $c \equiv 0 \pmod{24}$ we can show:

Theorem 3 *A vertex operator superalgebra V of central charge $c = 24$, 48, 72 or 96 with minimal weight $\mu(V) \geq c/24 + 1/2$ must be a vertex operator algebra.*

Proof. For a vertex operator superalgebra which is not a vertex operator algebra, the first coefficient of the character $\chi_{V'} = q^{-c/24} \sum_{n=0}^{\infty} B_n\, q^n$ vanishes since the conformal weight of the shadow is positive. This is impossible for $c = 24$. For the other values of c it is impossible to find values of the still undetermined a_i such that all B_n, $n \geq 1$, are nonnegative. □

The largest minimal weight of a vertex operator superalgebra V which is not a vertex operator algebra for the above four values of c is therefore $c/24 + 1/2$.

Problem: Is a self-dual vertex operator superalgebra V of central charge $c \equiv 0 \pmod{24}$ with minimal weight $\mu(V) = \left[\frac{c}{24}\right] + 1$ always a vertex operator algebra? (The proof in [Gau01] for lattices cannot directly be generalized since it uses a lower bound for the minimal norm of the shadow of an odd unimodular lattice whereas the analogous bound for vertex operator superalgebras is not obvious.)

The case of self-dual vertex operator superalgebras V of central charge 48 will be investigated further in Section 4.

Examples of vertex operator superalgebras achieving the lower bounds given in Table 1 can be constructed as follows:

For $c < 8$ and all other c with $\mu = \frac{1}{2}$ one uses $V_{\rm Fermi}^{\otimes 2c}$ where $V_{\rm Fermi}$ is the $c = \frac{1}{2}$ vertex operator superalgebra describing a single fermion.

For $8 \leq c \leq 23$ and $\mu = 1$ we use Thm. 5.3.2 [Höh95] and the table in Section 2 of [Höh97]. Both results depend on Schellekens classification of self-dual $c = 24$ vertex operator algebras and use the construction of Chapter 3 in [Höh95]. In particular, for integral c these examples are just the lattice vertex operator superalgebras associated to an integral unimodular lattice with minimal norm 2 or 3.

For $c = 23\frac{1}{2}$ the shorter Moonshine module $VB^\natural$ of [Höh95], Ch. 4, is an example with $\mu = 1\frac{1}{2}$ and for $c = 24$ the Moonshine module $V^\natural$ [FLM88] is an example with $\mu = 2$.

For $c = 32$, 40 and 48 the $\mathbf{Z}_2$-orbifolds of lattice vertex operator algebras associated to extremal even unimodular lattices in dimension 32, 40 and 48 [DGM90] are examples with $\mu = 2$. Since for $c = 32$ and 40 these vertex operator algebras have a Virasoro frame (cf. [DGH98]), one uses the same construction as for $VB^\natural$ to obtain self-dual vertex operator superalgebras of central charge $31\frac{1}{2}$ and $39\frac{1}{2}$ with $\mu = 1\frac{1}{2}$.

All remaining examples can be obtained by taking tensor products of two vertex operator superalgebras V and W of smaller central charge and using $\mu(V \otimes W) = \min\{\mu(V),\, \mu(W),\, 2\}$.

Table 1: Highest minimal weight μ of a self-dual vertex operator superalgebra of central charge c for $c \leq 48$

c	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8
μ	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
c	$8\frac{1}{2}$	9	$9\frac{1}{2}$	10	$10\frac{1}{2}$	11	$11\frac{1}{2}$	12	$12\frac{1}{2}$	13	$13\frac{1}{2}$	14	$14\frac{1}{2}$	15	$15\frac{1}{2}$	16
μ	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	1
c	$16\frac{1}{2}$	17	$17\frac{1}{2}$	18	$18\frac{1}{2}$	19	$19\frac{1}{2}$	20	$20\frac{1}{2}$	21	$21\frac{1}{2}$	22	$22\frac{1}{2}$	23	$23\frac{1}{2}$	24
μ	$\frac{1}{2}$	1	1	1	1	1	1	1	1	1	1	1	1	1	$1\frac{1}{2}$	2
c	$24\frac{1}{2}$	25	$25\frac{1}{2}$	26	$26\frac{1}{2}$	27	$27\frac{1}{2}$	28	$28\frac{1}{2}$	29	$29\frac{1}{2}$	30	$30\frac{1}{2}$	31	$31\frac{1}{2}$	32
μ	$\frac{1}{2}$–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	$1\frac{1}{2}$	2
c	$32\frac{1}{2}$	33	$33\frac{1}{2}$	34	$34\frac{1}{2}$	35	$35\frac{1}{2}$	36	$36\frac{1}{2}$	37	$37\frac{1}{2}$	38	$38\frac{1}{2}$	39	$39\frac{1}{2}$	40
μ	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–$1\frac{1}{2}$	1–2	1–2	1–2	1–2	1–2	1–2	1–2	1–2	1–2	1–2	1–2	$1\frac{1}{2}$–2	2
c	$40\frac{1}{2}$	41	$41\frac{1}{2}$	42	$42\frac{1}{2}$	43	$43\frac{1}{2}$	44	$44\frac{1}{2}$	45	$45\frac{1}{2}$	46	$46\frac{1}{2}$	47	$47\frac{1}{2}$	48
μ	1–2	1–2	1–2	1–2	1–2	1–2	1–2	1–2	1–2	1–2	1–2	1–2	1–2	$1\frac{1}{2}$–2	$1\frac{1}{2}$–2	2–3

3 Self-dual $N=1$ supersymmetric vertex operator superalgebras

We recall that an N=1 supersymmetric vertex operator superalgebra V is an vertex operator superalgebra V together with a superconformal element $\tau \in V_{3/2}$ such that the operators $G_{n+1/2} = \tau_{n+1}$ generate a representation of the Neveu-Schwarz superalgebra on V. This is the case precisely if $\tau_2\tau = \frac{2}{3}c\mathbf{1}$, $\tau_1\tau = 0$ and $\tau_0\tau = 2\omega$. The σ-twisted modules (or Ramond sectors) of V admit then a representation of the Ramond superalgebra. For a σ-twisted module one has under some unitary assumption that the conformal weight satisfies $h \geq \frac{c}{24}$ (cf. [LT89], p. 242).

Definition 4 The *minimal superconformal weight* $\mu^*(V)$ of an N=1 vertex operator superalgebra V is defined as the smallest degree of a highest weight vector of the N=1 super Virasoro algebra different from the vacuum vector. In case the only highest weight vector is the vacuum vector, we let $\mu^*(V) = \infty$.

One has

$$\chi_V = q^{-c/24}\left(\prod_{n=2}^{\infty}\frac{1+q^{n-1/2}}{1-q^n} + \prod_{n=1}^{\infty}\frac{1+q^{n-1/2}}{1-q^n}\left(\sum_{i\geq\mu^*(V)} P_i\cdot q^i\right)\right),$$

where P_i is the dimension of the space of highest weight vectors of degree i for the N=1 super Virasoro algebra.

For a self-dual N=1 supersymmetric vertex operator superalgebras V, the shadow V' is the unique σ-twisted V-module and hence has conformal weight $h(V') \geq c/24$. This implies that $\chi_{V'}$ has no pole in the cusp $i\infty$. By using the relation between the characters of V and V' given in the introduction it follows that χ_V has no pole in the cusp 1. Hence equation (3) gives

$$\chi_V = \sum_{r=0}^{k} a_r \chi_{1/2}^{2c-24r} = q^{-c/24}\sum_{n=0}^{\infty} C_n\, q^{n/2}$$

with $k = \left[\frac{c}{12}\right]$, function $\chi_{1/2}$ as in (4), and uniquely determined integers $a_0, \ldots, a_k$.

Denote by $\chi_{M_c^{N=1}} = q^{-c/24}(\prod_{n=2}^{\infty}\frac{1+q^{n-1/2}}{1-q^n})$ the character of the $N=1$ Viraso vertex operator superalgebra generated by τ. Following [Wit07] we make the following definition:

Definition 5 If the $a_0, \ldots, a_k$ are chosen such that one has

$$\chi_V = \chi_{M_c^{N=1}}\cdot(1 + A_{2k+1}\, q^{(k+1)/2} + A_{k+2}\, q^{(k+2)/2} + \cdots) \tag{13}$$

then (13) is called the *extremal (N=1 supersymmetric) character* for central charge c. A self-dual N=1 supersymmetric vertex operator superalgebra with this character is called an *extremal N=1 supersymmetric vertex operator superalgebra.*

It follows directly from the definitions that the minimal superconformal weight of a self-dual N=1 supersymmetric vertex operator superalgebra V satisfies $\mu^*(V) \geq \frac{1}{2}\left[\frac{c}{12}\right]+\frac{1}{2}$.

The character of the shadow V' of a self-dual N=1 supersymmetric vertex operator superalgebra V is

$$\chi_{V'} = \alpha \sum_{r=0}^{k} a_r \widetilde{\chi}_{1/2}^{2c-24r} = \alpha\, q^{c/12-[c/12]} \sum_{n=0}^{\infty} B_n\, q^n$$

with $\alpha = 1$ for integral and $\alpha = 1/\sqrt{2}$ for nonintegral values of c and $\widetilde{\chi}_{1/2}$ as in (6). If the a_i, $0 \leq i \leq k$, are chosen as in Definition 5 we call $\chi_{V'}$ the *extremal shadow character.*

Theorem 6 *A self-dual N=1 supersymmetric vertex operator superalgebra has minimal superconformal weight*

$$\mu^*(V) \leq \frac{1}{2}\left[\frac{c}{12}\right] + \frac{1}{2}$$

unless $c = 23\frac{1}{2}$ in which case $\mu^*(V) \leq 3/2$.

Proof. Let $k = [c/12]$ and $p = q^{1/2}$. We compute the coefficient A_{k+1} of the extremal character $\chi_V = \chi_{M_c^{N=1}} \cdot (1 + A_{k+1}\, q^{(k+1)/2} + \cdots)$ by the method used in the proof of Theorem 1 (see (9) and the following equation) and obtain that A_{k+1} is the coefficient of p^k in

$$-\frac{1}{k+1}\, p^{k+1} \cdot \chi_{1/2}^{24(k+1)-2c-1} \left[\chi'_{M_c^{N=1}} \chi_{1/2} - 2c \cdot \chi_{M_c^{N=1}} \chi'_{1/2}\right]. \qquad (14)$$

The coefficients of $p^{24(k+1)-2c}\chi_{1/2}^{24(k+1)-2c-1} = 1 + O(p)$ are nonnegative since $24(k+1) - 2c - 1 = 23 - 2c + 24[c/12] \geq 0$. Similar as in Lemma 2 it can be shown that all coefficients of $-p^{(2c+1)/24}(\chi'_{M_c^{N=1}}\chi_{1/2} - 2c \cdot \chi_{M_c^{N=1}}\chi'_{1/2})$ are positive besides the coefficient of p which is zero.

Thus A_{k+1} is positive unless $c = 23\frac{1}{2}$ in which case the extremal character is $q^{-47/48}(1 + 4372\, q^{3/2} + O(q))$ yielding $\mu^* \leq 3/2$. □

Remark: Witten defines extremal N=1 supersymmetric vertex operator superalgebras by only requiring that $\mu^* \geq \frac{1}{2}[c/12]$. Then the character is only determined up to the addition of a constant. The reason for this modification is that for c a multiple of 24 and $c \geq 96$ the first coefficient of the expansion of the extremal shadow character becomes negative, i.e., for those values of c an extremal N=1 supersymmetric vertex operator superalgebra cannot exist. Hence for those c the bound in Theorem 6 can be improved by $\frac{1}{2}$.

Numerical evidence suggests that besides for the mentioned case all coefficients of the extremal character and of the extremal shadow character are positive integral numbers.

Examples of extremal self-dual N=1 supersymmetric vertex operator superalgebras are $V_{\rm fermi}^{\otimes 2c}$ for $c = (3/2)k$, $k = 1, \ldots, 7$, the vertex operator superalgebra $V_{D_{12}^+}$ [Dun05] and the odd Moonshine module $VO^\natural$ [DGH88].

There exist other self-dual vertex operator superalgebras with the right extremal character who might have also the additional structure of an $N{=}1$ supersymmetric vertex operator superalgebra.

4 Central Charge $c = 48$ and the Monster

In this section, we study self-dual vertex operator algebras and vertex operator superalgebras of central charge 48 with large minimal conformal or superconformal weight. The question of possible monster symmetries of such vertex operator superalgebras is investigated.

We recall that Theorem 3 implies for central charge 48 that the minimal weight of a vertex operator superalgebra V which is not a vertex operator algebra can be at most $5/2$. More precisely, we can deduce from the requirement that the character of V and its shadow V' have nonnegative integral coefficients that for a vertex operator superalgebra with minimal weight $5/2$ there are only the following two possibilities for the characters:

$$\begin{aligned}\chi_V &= q^{-2} + 1 + 192512\, q^{1/2} + 21590016\, q + 863059968\, q^{3/2} + 20256751892\, q^2 + \cdots \\ \chi_{V'} &= q^{-1} + 1 + 42991892\, q + 40491808768\, q^2 + 8504047840194\, q^3 + \cdots \end{aligned} \tag{15}$$

$$\begin{aligned}\chi_V &= q^{-2} + 1 + 196608\, q^{1/2} + 21491712\, q + 864288768\, q^{3/2} + 20246003988\, q^2 + \cdots \\ \chi_{V'} &= 25 + 42991616\, q + 40491810816\, q^2 + 8504047828992\, q^3 + \cdots \end{aligned} \tag{16}$$

For lattices of dimension 48 and codes of length 48 an analogous result can be found in [HKMV05]. There exist lattices as well as codes realizing both possibilities for the theta series respectively weight enumerator. However, for vertex operator algebras the first case can be excluded.

Theorem 7 *Let V be a vertex operator superalgebra of central charge* 48 *with minimal weight $\mu(V) = 5/2$. Then the shadow of V has minimal conformal weight* 2*.*

Proof. Since $\mu(V) = 5/2$ is nonintegral, V is not a vertex operator algebra. Hence the even part $V_{(0)}$ has four irreducible modules $V_{(0)}$, $V_{(1)}$, $V_{(2)}$ and $V_{(3)}$, where $V = V_{(0)} \oplus V_{(1)}$ and $V' = V_{(2)} \oplus V_{(3)}$. The simple current extensions $W = V_{(0)} \oplus V_{(2)}$ and $\widetilde{W} = V_{(0)} \oplus V_{(3)}$ can be given the structure of self-dual vertex operator algebras and both are neighbours of V. Assume that the shadow of V has minimal conformal weight different from 2. Then the character of $V' = V_{(2)} \oplus V_{(3)}$ is given by equation (15). Thus either $V_{(2)}$ or $V_{(3)}$ has a one-dimensional degree 1 part; say $V_{(3)}$. In this case $\widetilde{W}_1$ is one-dimensional and generates a Heisenberg vertex operator subalgebra of central charge 1 with graded dimension $\prod_{n=1}^{\infty}(1-q^n)^{-1}$. This implies $\dim \widetilde{W}_2 \geq 2$ and hence $\dim \widetilde{W}_2 = 2$ using (15) again. A Heisenberg vertex operator algebra contains the one-parameter family $\omega_\lambda = \frac{1}{2}h_{(-1)}^2\mathbf{1} + \lambda h_{(-2)}\mathbf{1}$, $\lambda \in \mathbf{C}$, of possible Virasoro vectors, where ω_λ has central charge $1 - 12\lambda^2$. For the Virasoro element ω_{λ_0} of $\widetilde{W}$ one has therefore $\lambda_0 \neq 0$ since $\widetilde{W}$ has central charge 48. The vertex operator algebra $\widetilde{W} = V_{(0)} \oplus V_{(3)}$ admits a natural automorphism σ which acts by multiplication with $+1$ on $V_{(0)}$ and

with -1 on $V_{(3)}$. Since $\widetilde{W}_1 = \mathbf{C}h_{(-1)}\mathbf{1} \subset V_{(3)}$, it follows that $h^2_{(-1)}\mathbf{1}$ is in the $+1$-eigenspace of the involution σ. But ω_{λ_0} is also in the $+1$-eigenspace. This contradicts $\dim(V_{(0)})_2 = 1$.

Our assumption about the minimal conformal weight of the shadow to be different from 2 must therefore be wrong. □

If a vertex operator superalgebra with minimal weight $5/2$ and minimal conformal weight of the shadow be 1 would exist, then the neighbour vertex operator algebra W as in the proof would be an example of an extremal vertex operator algebra, i.e., a self-dual vertex operator algebras with minimal weight $\mu = 3$.

It follows from Theorem 6 or the discussion in [Wit07], Sect. 3.2, that the minimal superconformal weight of an N=1 supersymmetric vertex operator superalgebra V of central charge 48, is at most $5/2$. If V is extremal, the characters of V and V' are ([Wit07], eq. (3.60)):

$$\begin{aligned}
\chi_V &= q^{-2} + q^{-1/2} + 1 + 196884\, q^{1/2} + 21493760\, q + 864299970\, q^{3/2} \\
&\qquad + 20246053140\, q^2 + 333202640600\, q^{5/2} + 4252023300096\, q^3 + \cdots, \\
\chi_{V'} &= 1 + 42987520\, q + 40491712512\, q^2 + 8504046600192\, q^3 + \cdots. \qquad (17)
\end{aligned}$$

Theorem 8 (cf. [Wit07], discussion at the end of Sect. 3.3) *Let V be an extremal self-dual N=1 supersymmetric vertex operator superalgebra of central charge 48. Then V has an extremal self-dual vertex operator algebra W as neighbour.*

Proof. As in the proof of Theorem 7, let $W = V_{(0)} \oplus V_{(2)}$ and $\widetilde{W} = V_{(0)} \oplus V_{(3)}$ be the two vertex operator algebra neighbours of V. From the character of $V' = V_{(2)} \oplus V_{(3)}$ given in equation (17) it follows that either $V_{(2)}$ or $V_{(3)}$, say $V_{(2)}$, has zero-dimensional degree 2 part. Then $V_{(2)}$ has minimal conformal weight at least 3 and $W = V_{(0)} \oplus V_{(2)}$ has minimal weight 3, i.e., is an extremal vertex operator algebra of central charge 48. □

Remark: The same argument shows that a neighbour of an extremal self-dual N=1 supersymmetric vertex operator superalgebra of central charge 72 is an extremal vertex operator algebra. The characters of V and V' of such a vertex operator superalgebra can be found in [Wit07], Appendix A.

Witten observes that the first coefficients of the modular functions in (17) are simple linear combinations of dimensions of irreducible representations of the monster simple group and further that such a decomposition is compatible with the N=1 super Virasoro algebra module structure of V (see [Wit07], eq. (3.61) and (3.62)).

Since any action of the monster group on a central charge 48 vertex operator superalgebra V induces such an action on the $V_{(0)}$-modules $V_{(i)}$, $i = 0, 1, 2, 3$, and hence on the vertex operator algebra neighbours W and $\widetilde{W}$ (it is easy to see that this action respects the vertex operator algebra structure), we have the following corollary to Theorem 8:

Corollary 9 *Assume that V is an extremal self-dual N=1 vertex operator superalgebra of central charge 48 on which the monster acts by automorphisms. Then the*

monster acts also on the extremal neighbour vertex operator algebra W as in Theorem 8 by automorphisms such that the actions coincides on the common vertex operator subalgebra $V_{(0)}$ of V and W.

Witten asks in [Wit07], Sect. 3.1, if extremal self-dual vertex operator algebras of central charge $c = 24k$ exist for all natural numbers k, if they are unique, and if they have a monster symmetry. In the following, it is shown that for $c = 48$ at least a monster symmetry is impossible under certain natural assumptions.

The character of a self-dual vertex operator algebra of central charge divisible by 24 is a polynomial with integer coefficients in the modular invariant $j = q^{-1} + 744 + 196884\,q + 21493760\,q^2 + \cdots$ or, equivalently, in $J = j - 744$, the character of the moonshine module $V^\natural$. In particular, for the character of an extremal vertex operator algebra W of central charge 48 one has (see [Höh95], Table 5.1):

$$\begin{aligned}\chi_W &= J^2 - 393767 \qquad (18)\\ &= q^{-2} + 1 + 42987520\,q + 40491909396\,q^2 + 8504046600192\,q^3 + \cdots.\end{aligned}$$

If we assume that the monster acts on W nontrivially by vertex operator algebra automorphisms, the simplest way to consider W as a module for the monster is to assume that one has

$$W = V^\natural \otimes V^\natural - (2R_2 + R_1) \qquad (19)$$

as graded monster modules, where R_1 denotes the trivial 1-dimensional and R_2 is the irreducible 196883-dimensional representation of the monster. This also guarantees that the monster module structure is compatible with the Virasoro module structure of W as one can easily check. For an element g in the monster the graded trace of g acting on W is then given by

$$\mathrm{tr}(g|W) = q^{-2}\sum_{n=0}^{\infty} \mathrm{tr}(g|W_n)\,q^n = T_g^2 - (2\,\mathrm{tr}(g|R_2) + 1), \qquad (20)$$

where T_g is the McKay-Thompson series of g, i.e., the graded trace of g acting on $V^\natural$.

Furthermore, if we assume that the monster module structure of the first few homogeneous spaces of a $N = 1$ supersymmetric vertex operator superalgebra V of minimal superconformal weight $5/2$ is the one given in [Wit07], eq. (3.61) and (3.62), then the monster module structure of the extremal neighbour of V as in Corollary 9 is also compatible with (19) at least if we modify the monster module structure of [Wit07] eq. (3.61) and (3.62) by exchanging $R_1 + R_4 + R_5$ with $R_3 + R_6$ if necessary. (It was remarked in [Wit07] that such modifications are possible.) More precisely, we could assume that for V one has

$$\begin{aligned}V_0 &= R_1\\ V_{1/2} &= 0\\ V_1 &= 0\\ V_{3/2} &= R_1\\ V_2 &= R_1\\ V_{5/2} &= R_1 + R_2\end{aligned}$$

$$
\begin{aligned}
V_3 &= R_1 + R_2 + R_3 \\
V_{7/2} &= 2R_1 + 2R_2 + R_3 + R_4 \\
V_4 &= 4R_1 + 4R_2 + R_3 + 2R_4 + R_5 \\
V_{7/2} &= 5R_1 + 5R_2 + 2R_3 + 3R_4 + 2R_5 + R_7 \\
V_5 &= 5R_1 + 7R_2 + 4R_3 + 4R_4 + 2R_5 + 2R_6 + R_7 + R_8
\end{aligned}
$$

and for V' one has

$$
\begin{aligned}
V'_0 &= 0 \\
V'_1 &= 0 \\
V'_2 &= R_1 \\
V'_3 &= 2 \times (R_1 + R_2 + R_3) \\
V'_4 &= 2 \times (2R_1 + 3R_2 + 2R_3 + R_4 + R_6) \\
V'_5 &= 2 \times (3R_1 + 7R_2 + 6R_3 + 2R_4 + 4R_6 + R_7 + R_8),
\end{aligned}
$$

where R_i denotes the i-th representation of the monster. This decomposition remains compatible with an N=1 super Virasoro algebra module structure.

We will also use the following conjecture about the structure of $\mathbf{Z}_2$-orbifolds of self-dual vertex operator algebras which seems not to be completely proven:

Conjecture 10 *Let t be an involutive automorphism of a self-dual vertex operator algebra W. Then the fixpoint vertex operator subalgebra $W^{\langle t\rangle}$ is rational and has the fusion algebra $\mathbf{Z}[\mathbf{Z}_2 \times \mathbf{Z}_2]$. The conformal weights of the four isomorphism types of irreducible $W^{\langle t\rangle}$-modules are either congruent to 0, 0, 0, $1/2 \pmod 1$ (case I) or 0, 0, $1/4$, $3/4 \pmod 1$ (case II).*

For W the self-dual lattice vertex operator algebra V_{E_8} of central charge 8 with $E_8(\mathbf{C})$ as automorphism group, the two conjugacy classes of involutions of $E_8(\mathbf{C})$ realize both cases I and II. For W the moonshine module $V^\natural$, the two conjugacy classes of involutions in the monster correspond both to case I.

In case I, one can extend $W^{\langle t\rangle}$ by the module of conformal weight $1/2 \pmod 1$ to obtain a self-dual vertex operator superalgebra as neighbour of W.

Theorem 11 *The monster cannot act by automorphisms on an extremal self-dual vertex operator algebra W of central charge 48 such that as a graded monster module one has $W = V^\natural \otimes V^\natural - (2R_2 + R_1)$ provided Conjecture 10 holds.*

In fact, we only will need that the monster module structure of W is the stated one for W_n, $0 \leq n \leq 5$.

Proof. Let t be an involution in the monster which has a twofold cover of the baby monster as centralizer, i.e., an involution of type $2A$ in atlas notation. Then the character of the fixpoint vertex operator algebra $W^{\langle t\rangle}$ is

$$
\begin{aligned}
\chi_{W^{\langle t\rangle}} &= \frac{1}{2}\left(\chi_W + \mathrm{tr}(t|W)\right) \\
&= q^{-2} + 1 + 21590016\, q + 20256751892\, q^2 + 4252454830080\, q^3 + \cdots
\end{aligned}
$$

where we have used equations (18), (20), the character value $\mathrm{tr}(t|R_2) = 4371$ and the q-expansion of the Thompson series T_t as conjectured in [CN79] and proven in [Bor92]. By comparing this expansion with the one in (15) it follows that $\chi_{W^{\langle t\rangle}} = \chi_{V_{(0)}}$ where $V_{(0)}$ is the even vertex operator subalgebra of a vertex operator superalgebra of central charge 48 with minimal weight $\mu(V) = 5/2$ and shadow of minimal conformal weight 1, because both functions are modular functions for $\Gamma_0(2)$. The width of $\Gamma_0(2)$ in its other cusp is 2 and hence $\chi_{W^{\langle t\rangle}}(-1/\tau)$ has an expansion in powers of $q^{1/2}$. This implies that t is an involution in $\mathrm{Aut}(W)$ belonging to case I of Conjecture 10: Denoting the four irreducible modules of $W^{\langle t\rangle}$ by M_i, $i = 1, \ldots, 4$, the expansion of $\chi_{W^{\langle t\rangle}}(-1/\tau) = \frac{1}{2}\sum_{i=1}^{3}\chi_{M_i}(\tau)$ would contain in case II non-even powers of $q^{1/4}$. Since we are in case I, we can extend the vertex operator algebra $W^{\langle t\rangle}$ to a self-dual vertex operator superalgebra V. From the explicit expansion of $\chi_{W^{\langle t\rangle}}$ in the other cusp, we see that the characters of V and its shadow are the ones given in (15). Thus V has minimal weight $5/2$ and the shadow of V has minimal weight 1. However, by Theorem 7 such a vertex operator superalgebra cannot exist. □

Remark: If we take instead of an involution of type $2A$ an involution of type $2B$ in the monster, then the corresponding neighbour vertex operator superalgebra has the character given in (16).

Theorem 11 together with Corollary 9 implies that under reasonable assumptions it is impossible for the monster simple group to act on an extremal N=1 supersymmetric vertex operator superalgebra in the way suggested in [Wit07].

If we would have used in Corollary 9 for V exactly the monster module structure as in [Wit07], then the character of the constructed vertex operator superalgebra neighbour of W would have non-integral coefficients.

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